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STRATEGIC BEHAVIOR AND A NOTION OF EX ANTE EFFICIENCY IN A VOTING MODEL

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Summary:

A person is said to prefer in the stochastic dominance sense one lottery over outcomes over another lotter over outcomes if the probability of his (at least) first choice being selected in the first lottery is greater than or equal to the analogous probability in the second lottery, the probability of his at least second choice being selected in the first lottery is greater than or euqal to the analogous probability in the second lottery, and so on, with at least one strict inequality. This (partial) preference relation is used to define straightforwardness of a social choice function that maps profiles of ordinal preferences into lotteries over outcomes. Given a prior probability distribution on profiles this partial preference ordering (taking into account the additional randomness) is used to induce a partial preference ordering over social choice functions for each individual. These are used in turn to define ex ante Pareto undominated (efficient) social choice functions. The main result is that it is impossible for a social choice function to be both ex ante efficient and straightforward. We also extend the result to cardinal preferences and expected utility evaluations.

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I. Introduction

The Arrowian social choice problem consists of constructing a "nice" mapping (constitution) which associates to every configuration of individual preferences over a set of alternatives (profile) a social preference ordering over the same set of alternatives. Within this framework there is an analogous problem with which we deal, that of constructing a mapping (social choice function) which associates to every profile an outcome rather than a preference ordering. One particularly nice property that we would like in a social choice function, is that if an individual is required to supply information about his preferences in order to effectuate the social choice function, misrepresenting his preferences will not be to his advantage. This emphasis in examining the strategic considerations in implementing a social choice function is due to Farquharson [1969]. Gibbard [1973] and Satterthwaite [1975] proved that in general any social choice function which is not dictatorial is manipulable, i.e., for some profiles misrepresentation of some agent's preferences will be to his benefit. If we allow lotteries over alternations to be assigned by social choice functions, the above result breaks down. Consider the social choice function which assigns to every profile which assigns equal probability to each person's first choice. This system is neither dictatorial nor manipulable.

This system has a further property which was one of the desideratar postulated by Arrow for the deterministic social welfare function framework. That is, given a profile, an alternative selected by this procedure (including the resolution of the lottery) is not unanimously dominated by any other alternative. In the usual terminology this should

be called ex post Pareto efficiency, which does not in general imply ex ante efficiency. When we speak of ex ante efficiency of a constitution or social choice function which is applicable to all profiles (Arrow's postulate of unrestricted domain), we mean not before the resolution of the lottery but before the profile itself is known. Hence, the distinction between ex post and ex ante efficiency in social choice is not restricted to randomized procedures, but applies to the deterministic framework as well. Arrow's justification of the postulate of unrestricted domain is precisely that the decision making process should be applicable to all possible profiles since when we choose it, we don't know to which profiles it will be applied.

Arrow's Pareto principle, which is ex post, should be implied by a reasonable notion of ex ante efficiency. Thus we are suggesting that ex post efficiency is necessary but not sufficient as a proper concept of efficiency of a constitution. Since Arrow proved an impossibility theorem, the use of the weaker condition leads in fact to a stronger result.

Our main aim is to show the incompatibility of straightforwardness and efficiency when lotteries are permitted. Since, as mentioned earlier, the random dictator random choice function is both straightforward and ex post efficient, we must use the stronger notion of ex ante efficiency. One of the contributions of this paper is to introduce a very weak notion of ex ante efficiency, one which can be applied to the ordinal framework. This notion is that of comparison by stochastic dominance and is the subject of the next section. We should note that this basis of comparison circumvents the problem of interpersonal comparison of utility. In the following section we use the same technique of stochastic dominance to

define straightforwardness and prove the main impossibility theorem. In the last section we discuss cardinal preferences and show that the main result carries over to the cardinal structure.

II. Stochastic Dominance and Pareto Efficiency

There is a fixed finite set of mutually exclusive alternatives A. $\hat{A}$ is the set of probability distributions (or lotteries) over A. We identify A with the subset of $\hat{A}$ of degenerate probability distributions. N denotes the nonempty finite set of persons each of whom has a complete, transitive, asymmetric binary (preference) relation P over A. $\tilde{P}$ is the set of all such relations and $\tilde{P}^N$ is the set of all N-lists of preference relations. We will generally call these N-lists profiles and denote a profile by $\underline{P} \equiv (P_i)_{i\varepsilon N}$. S will be the (nonempty) strategy set common to all persons. A random outcome function (ROF) is a function f: $S^N \to \hat{A}$. An element in S^N will be called a selection and denoted $\underline{s} \equiv (s_i)_{i\varepsilon N}$. As usual a choice function is a mapping from profiles to outcomes. We will be concerned with random choice functions (RCF) which take profiles to random variables over outcomes, F: $\tilde{P}^N \to \hat{A}$.

In a deterministic framework a (deterministic) choice function f: $\tilde{P}^N \to A$ is said to be Pareto efficient (PE) is for every profile $\underline{P} \varepsilon \tilde{P}^N$, F($\underline{P}$) is Pareto efficient in A in the usual sense with respect to $\underline{P}$, i.e., for any a ε A: $[(\exists\ i\varepsilon N \text{ such that } aP_iF(\underline{P})) \Rightarrow (\exists\ j\varepsilon N \text{ such that } F(\underline{P})\ P_j a)]$. In other words the usual definition of efficiency first defines efficiency of an outcome at a profile and then a choice function is called efficient if it is efficient for all profiles. This concept is the one introduced into social choice theory by Arrow [1963] and is consistent with the notion of Pareto efficiency in the Bergson-Samuelson social welfare function and Hurwicz's [1972] notion of non-wastefulness of allocation mechanisms.

But consider the design of a social choice function which is to be applied to many situations (i.e., profiles) or to unknown situations (because preferences are fundamentally unknown to a planner). In this case this notion of pointwise efficiency seems a necessary but not sufficient criterion of efficiency to apply to choice functions. This implicit stochastic nature of the profile was generally disregarded in the deterministic choice function framework. However, when randomness enters in the choice function (i.e., an "outcome" associated with a specific profile is a random variable) one must use stochastic considerations in the welfare evaluation of the choice function. It then becomes compelling to consider not only the stochastic nature of the outcome for a specific profile but the stochastic nature of the profiles themselves.

Formally we assume there is a list of probability distribution over profiles, one for each person. <u>We assume that the support of each distribution is the entire set of profiles and we make a symmetry across persons assumption</u>. Specifically we assume that if a profile is obtained from another profile by a permutation of persons, then the probabilities of the profiles are equal. <u>We also restrict our attention to symmetric (across persons) choice and outcome functions</u>. That is for an RCF F, $F(\underline{P}) = F(\underline{P}')$ whenever $\underline{P}'$ is a permutation (across people) of $\underline{P}$; analogously for random outcome functions. While the assumption of symmetric priors is essential for our main result, the symmetry assumption on RCF's is for expository purposes only. We will comment further on this at the end of section three.

We will now make explicit our notion of efficiency of an RCF. Consider two RCFs, F and F'. F is said to Pareto dominate F' if each

person prefers F to F' taking into account the probability distribution over the outcomes for each profile as well as his probability distribution over profiles. We say that an RCF F is Pareto efficient if it is not Pareto dominated by any RCF F'.

If $\underline{\pi}$ is the list of the persons' probability distributions over profiles, this domination relationship depends on $\underline{\pi}$. Thus the Pareto efficiency should properly be called $\underline{\pi}$-Pareto efficiency. We will never be considering two different lists of probability distributions over profiles simultaneously; hence we will refer only to Pareto efficiency and omit $\underline{\pi}$.

What has been left unstated until now is how a person compares two RCFs, given his probability distribution π over profiles. Since the profiles are of ordinal preferences (i.e., in any realization of π a person expects to have ordinal preferences over the outcomes) stochastic dominance will be the proper tool in comparing RCFs. As mentioned in the introduction expected utility as a tool for comparison will be treated in the sequel (after introducing a notion of cardinal profiles).

We remind the reader that given two k-lists of numbers (in our case probability distributions) $q = (q^i)_{i=1}^k$, $r = (r^i)_{i=1}^k$ we say that q stochastically dominates r if for all j = 1, ..., k $\Sigma_{i=1}^j q^i \geq \Sigma_{i=1}^j r^i$ with equality for j = k and at least one strict inequality. We will denote this relation by q SD r. An RCF F and a probability distribution over profiles induce a k-list $q = (q^i)_{i=1}^k$ where $k = \#A$ and q^i is the probability of a person's i^{th} ranked alternative being chosen. Given two RCFs, F and F', it is these induced probability vectors which are compared via stochastic dominance to determine the ranking of F and F'.

We will write FSDF' [or FSD(π)F'] if the corresponding probability vectors, q and r, stand in this relation. When we consider x, y $\epsilon \hat{A}$ we write x SD y if the induced probability vectors q_x and q_y stand in the same relation.

To illustrate this concept we will compare two RCFs, F and G, with three persons and three alternatives. F is the random dictator; for any profile $\underline{P}$, F($\underline{P}$) is the probability distribution over A which assigns probability 1/3 to each person's first choice. G is the simple majority RCF with a tie breaking rule of assigning probability 1/3 to each alternative in the cyclic profiles. Each person's probability distribution over profiles is uniform.

We will now calculate the probability vector q induced by F. Let us consider any person, say 1. The probability that he is the random dictator is 1/3 and in this case he surely gets his first choice. The probability that the second person is the random dictator times the probability that the second peron's first choice coincides with that of 1 is $1/3 \cdot 1/3 = 1/9$. Similarly the probability that the third person is the random dictator and that his first choice coincides that of 1 is 1/9. Thus q^1, the probability that 1 receives his first choice is $1/3 + 1/9 + 1/9 = 5/9$. If person 1 is the random dictator he will not receive his second choice. If 2 (or 3) is the random dictator, the probability that 1's second choice is the first choice of 2 (or 3) is 1/3. Thus $q^2 = 1/3 \cdot 1/3 + 1/3 \cdot 1/3 = 2/9$. Similarly $q^3 = 2/9$.

Next we will calculate the probability vector r induced by G. Person 1's top ranking alternative can be chosen either as a majority winner or under the tie-breaking rules. The probability of a tie, i.e.,

of a cyclic profile is 1/18: In that case the probability of 1's top ranked alternative being chosen is 1/3. In the case of a majority winner there is probability 1/9 of a unanimous first choice. There is probability 2/3 that exactly two persons have the same first choice and probability 2/3 that one of them is person 1.

The probability that the first choices of all three persons are distinct is 2/9. Included in these profiles are the cyclic profiles which have probability 1/18. Thus the probability is 3/18 that all three persons have distinct first choices and there is a majority winner. Hence the probability that 1's first choice is chosen in this case is $1/3 \cdot 3/18 = 1/18$. Summing $r^1 = 1/3 \cdot 1/18 + 1/9 + 2/3 \cdot 2/3 + 1/18 = 34/54$.

The probability that 1's third choice is chosen is $1/3 \cdot 1/18$ under the tie-breaking rule and 1/9 if there is a majority winner (this can occur only if persons 2 and 3 have 1's third ranked alternative as their common top ranked alternative). Thus $r^3 = 1/3 \cdot 1/18 + 1/9 = 7/54$; hence $r^2 = 13/54$. The conscientious reader will compute r^2 directly to verify this. It is clear that r stochastically dominates q, hence for each person G SD F, i.e., G Pareto dominates F.

The relative inefficiency of the random dictator compared to majority rule can be seen clearly by examining the following profile (and permutations of this profile across people). The columns represent the individual preference orderings.

x	z	y
y	x	x
z	y	z

If we look only at the top ranked alternatives the random dictator seems perfectly reasonable. But looking at the entire preference profile, and

if a person takes into account that he is equally likely to be in any position in the profile he should favor x over z or y. Majority rule takes these considerations into account and the notion of stochastic dominance captures the advantage of majority fule over random dictator for such profiles without needing any cardinality of the preferences. It should be noted that for this example (i.e., 3 persons, 3 alternatives) the fact that majority fule Pareto dominates the random dictator does not depend on their being a common prior probability distribution. The result holds regardless of the persons'probability distributions so long as they satisfy our symmetry assumption. The discussion which follows examines this phenomenon more fully.

If we observe, as in the profile above, that an RCF F is "inefficient" on a specific subset of profiles, it must be dominated. To formalize this consider a set of profiles $\Gamma \subset \tilde{P}^N$ where Γ is closed under permutations across persons. That is if $\underline{P} \varepsilon \Gamma$ and $\underline{P}'$ is a permutation across persons of $\underline{P}$, then $\underline{P}' \varepsilon \Gamma$. We refer to such a set as a <u>symmetric subset of profiles</u>. We say that an RCF F stochastically dominates an RCF G on Γ if F SD(ϕ) G where $\phi = \pi/\Gamma$, i.e., ϕ is π conditioned on Γ. If Γ is the symmetric span of a single profile, then ϕ is identical for all persons due to the assumed symmetry of each person's probability distribution over $\tilde{P}^N$.

<u>Lemma</u>: Given an RCF F and a symmetric subset of profiles Γ, if F is stochastically dominated on Γ, then F is stochastically dominated.

<u>Proof</u>: Suppose F is stochastically dominated by G on Γ. Define a new RCF H such that $H(\underline{P}) = F(\underline{P})$ for $\underline{P} \notin \Gamma$ and $H(\underline{P}) = G(\underline{P})$ for $\underline{P} \varepsilon \Gamma$. We

will show that H SD F. Let us denote by q the probability distribution induced by F and π, i.e., q^i is the probability under π that a person's i^{th} ranked alternative is chosen by F. Similarly denote by r the distribution induced by H and π, by s the distribution induced by F and $\pi|\Gamma$, by t the distribution induced by G and $\pi|\Gamma$, and finally by v the distribution induced by F and $\pi|(\tilde{P}^N\setminus\Gamma)$. Thus the distribution induced by H and π, r satisfies

$$r = \pi(\tilde{P}^N\setminus\Gamma)\ v + \pi(\Gamma)t$$

and similarly

$$q = \pi(\tilde{P}^N\setminus\Gamma)\ v + \pi(\Gamma)s$$

Note that by hypothesis t SD s. The lemma then follows from the following two obvious claims.

<u>Claim 1</u>: For any two vectors a and b and a positive number α: a SD b $\Longleftrightarrow$ αa SD αb.

<u>Claim 2</u>: For any three vectors a, b and c: a SD b $\Longleftrightarrow$ a + c SD b + c.

Q.E.D.

The claims in effect say that the relation SD over vectors is invariant under positive linear transformations.

We made the assumption of positive probability of each profile to make the conditional probabilities well defined and consistent with claims 1 and 2. The symmetry of Γ was used to guarantee that the RCF H is symmetric across persons.

It should be noted that a simple corollary of this lemma is that if an RCF is Pareto dominated on a symmetric set of profiles, then it is Pareto dominated. The only restriction on the probability distributions over profiles of different persons is that of symmetry and positivity.

The converse of the lemma is false. That is, an RCF F which is undominated on every component of some partition of $\tilde{P}^N$ into symmetric subsets may be dominated on $\tilde{P}^N$. To see this we present the following example.

Let Γ_1 and Γ_2 be the symmetric spans (across persons) of the following profiles

$$\begin{array}{cccc} & \Gamma_1 & & \\ x & x & z & y \\ z & z & y & z \\ y & y & x & x \end{array} \qquad\qquad \begin{array}{cccc} & \Gamma_2 & & \\ z & z & z & y \\ y & y & y & x \\ x & x & x & z \end{array}$$

By the symmetric span of a profile we mean all the profiles obtained from it by permutations across persons. Let F be an RCF which chooses x for any profile in Γ_1, y for any profile in Γ_2 and is undominated on the set of all other profiles. We see that F is undominated on Γ_1 and Γ_2 as well. For instance, q_x on Γ_1 is (1/2, 0, 1/2) and there is no other random outcome which gives as high a probability of getting a person's first choice. In a similar manner choosing y for profiles in Γ_2 maximizes the probability of having a person's first or second choice, hence is

undominated. But q_z on $\Gamma_1 \cup \Gamma_2$ is (1/2, 3/8, 1/8) whereas F on $\Gamma_1 \cup \Gamma_2$ induces the vector 1/2(1/2, 0, 1/2) + 1/2(1/4, 3/4, 0) = (3/8, 3/8, 1/4). Thus we see that an RCF G which chooses z on $\Gamma_1 \cup \Gamma_2$ and coincides with F off $\Gamma_1 \cup \Gamma_2$ stochastically dominates F.

This same example shows clearly how the Pareto dominance relation on RCFs depends on the probability distribution π over profiles. In the above example if $\pi(\Gamma_2)/\pi(\Gamma_1)$ is nearly zero or infinity, q_z would not stochastically dominate the probability weighted average of q_x and q_y.

Another example which will be used to prove a main result involves the symmetric span of the following four person, four alternative profile.

$$(*)\qquad \begin{array}{cccc} x & y & z & w \\ y & x & x & y \\ z & w & y & z \\ w & z & w & x \end{array}$$

For this set of profiles if y is chosen the conditional (on this set of profiles) probability vector induced is q_y = (1/4, 2/4, 1/4, 0). Similarly if x is chosen the vector induced is q_x = (1/4, 1/2, 0, 1/4), if z is chosen the induced is q_z = (1/4, 0, 2/4, 1/4), and if w is chosen the induced vector is q_w = (1/4, 0, 1/4, 2/4). We see that q_y stochastically dominates each of the vectors q_x, q_z and q_w. Thus it is clear that q_y dominates any other vector which is induced by a random outcome which assigns positive probability to any alternative other than y. Hence by the lemma any undominated RCF must choose the alternative y for this subset of profiles.

Examination of these examples suggests how one might construct an undominated RCF for any number of persons and alternatives. In

particular suppose one defines an RCF such that for the symmetric span of any profile the outcome chosen for this subset of profiles induced a vector probabilities which maximizes the probability of a person getting his first choice and subject to this condition maximizes the probability of a person getting his second choice, and so on. This leads us to the plurality (or lexicographic pluralility) choice function which is formalized as follows:

<u>Theorem</u>: Given a profile $\underline{P} \varepsilon \tilde{P}^N$ let $F(\underline{P})$ be the lottery which assigns equal probability to each alternative in

$$B = \{x \varepsilon A \mid \forall y \varepsilon A: \;\; [(\;\; i \text{ s.t. } m_i(y) > m_i(x)) \Rightarrow (\;\; j < i \text{ s.t. } m_j(y) < m_j(x))]\}$$

where $m_k(z)$ is the number of persons for whom z is the k^{th} ranked alternative in $\underline{P}$. Then F is (Pareto) stochastically undominated.

<u>Proof</u>: Consider any person. We remind the reader that his underlying probability distribution over profiles π is symmetric across persons. Suppose that there is an RCF G which stochastically dominates F. Let q and r be the probability vectors induced by F and G, respectively. If r SD q then there exists k such that for $j < k$, $\Sigma_{i=1}^{j} r^i = \Sigma_{i=1}^{j} q^i$ and $\Sigma_{i=1}^{k} r^i > \Sigma_{i=1}^{k} q^i$. Thus $r^k > q^k$. Let $(\Gamma_\delta)_{\delta \varepsilon \Delta}$ be the partition of $\tilde{P}^N$ such that each Γ_δ is the symmetric span of a single profile.

We see then that

$$q = \Sigma_{\delta \varepsilon \Delta} \, \pi(\Gamma_\delta) \, q_\delta$$

and

$$r = \Sigma_{\delta \varepsilon \Delta} \, \pi(\Gamma_\delta) \, r_\delta$$

where q_δ is the probability vector induced by F on the symmetric span Γ_δ. Similarly r_δ is the probability vector induced by G on the symmetric span Γ_δ. Consider the first i, say e, such that for some $\delta\varepsilon\Delta$, $q_\delta^i \neq r_\delta^i$. We will show that the inequality must be of the form $q_\delta^e > r_\delta^e$. By the decomposition of q and r the last inequality and the equality for all lower indices implies that $q^i = r^i$ for $i < e$ and $q^e > r^e$. This is a contradiction to the negation assumption that G SD F, i.e., r SD q.

The above inequality is a simple consequence of the definition of lexicographic plurality. Indeed, F and G are each constant on every Γ_δ (since by assumption we are treating only symmetric choice functions). If $q_\delta \neq r_\delta$ and $\underline{P}\varepsilon\Gamma_\delta$, $G(\underline{P})\varepsilon\hat{A}$ is not a probability distribution over B (it is clear that B as in the statement of the theorem is identical for all $\underline{P}$ in a fixed Γ_δ). This is because if x, y ε B, $q_x = q_y = q_\delta$. Thus $G(\underline{P})$ must assign positive probability to some z $\not\varepsilon$ B.

For any z $\not\varepsilon$ B and any x ε B for the first i such that $m_i(z) \neq m_i(x)$ $m_i(z) < m_i(x)$. Or equivalently $q_z^i < q_x^i$. Since r_δ is the vector of probabilities induced by $G(\underline{P})$ with support not included in B, for the i such $r_\delta^i \neq q_\delta^i$ and $r_\delta^i < q_\delta^i$.

Since this must hold for each person, the result obtains.

Q.E.D.

We end this section by observing that the theorem above illustrates how one could construct other undominated RCFs. Lexicographic plurality maximizes the partial sums in the lexicographic order. However, we can get a stochastically undominated vector if we maximize the partial sums

in any order, if $j_1, \ldots, j_k$ is a permutation of $1, \ldots, k$ maximize the sum of the first j_1 coordinates, then maximize the sum of the first j_2 coordinates subject to the first maximization, and so on.

III. Straightforwardness in the Sense of Stochastic Dominance

We turn our attention now to the implementation of an RCF. By the implementation of an RCF we mean the design of an ROF and a solution concept, so that for every profile the resulting solution is precisely that random outcome which was prescribed by the RCF. Our goal here is to examine the possibility of implementation where the solution concept is dominant strategy equilibria. Given an ROF f: $S^N \to \hat{A}$ and a $\underline{P} \in \tilde{P}^N$ a selection $\underline{s}^* = (s_i^*)_{i\varepsilon N}$ is said to be dominant strategy equilibrium if for all $j\varepsilon N$ and $\underline{s} = (s_i)_{i\varepsilon N}$, $f(\underline{s})$ does not stochastically dominate $f\left((s_i)_{i\neq j}, s_j^*\right)$. As usual the ROF f is said to be straightforward (in the stochastic dominance sense) if there exists a dominant strategy equal for each profile. The plausibility of dominant strategy equilibria and straightforward outcome functions is well known (see Farquharson [1969] or Gibbard [1973]). A main difference between their treatment of straightforwardness and ours is that they restrict comparisons to deterministic outcomes using the preferences P_i while we allow comparisons of random outcomes, still using the preferences P_i on A. Clearly the stochastic dominance relation restricted to A (relative to P_i) coincides with P_i. Since the SD relation over $\hat{A}$ is not complete, for a given profile we may have that there are not only multiple dominant strategy equilibria but non-unique equilibrium outcomes as well. Given an RCF F we say that an ROF f implements F if for all $\underline{P} \in \tilde{P}^N$ $F(\underline{P}) \in \{f(\underline{s}) | \underline{s}$ is a dominant strategy equilibrium for $\underline{P}\}$. If we construct a mapping w: $\tilde{P}^N \to S^N$ such that $w(\underline{P}) \in \{\underline{s} | \underline{s}$ is a dominant strategy equilibrium and $f(\underline{s}) = F(\underline{P})\}$ the composition of f and w is a straightforward ROF with preference relations as strategies which implements F and furthermore, this composition

as a function from the set of profiles into outcomes coincides with F. Thus if an RCF F can be implemented at all it can be implemented by itself (as an ROF where the strategies are the individual preferences).

An example of an RCF which is straightforward (i.e., implemented by itself) is that of majority rule with a special tie-breaking rule.

Theorem: Given a profile $\underline{P} \in \tilde{P}^N$ with #N odd let $F(\underline{P})$ be the lottery which assigns probability 1 to alternative x if x is a majority winner and $F(\underline{P})$ be the lottery which assigns equal probability to all outcomes if there is no majority winner. Then F is straightforward.

Proof: We consider first the case where there is a majority winner x for a profile $\underline{P}$. Consider an individual j. Suppose by change of strategy, i.e., misrepresentation of his preferences, he can change the outcome.

Then the new outcome is either a new majority winner or the uniform probability over alternatives. If there is a new majority winner y we see that with his correct preferences a majority prefers x to y and with his revised preferences a majority prefers y to x. Hence the j^{th} person must have preferred x to y in his true preferences.

If there is not a new majority winner we note that the uniform distribution over alternatives stochastically dominates the outcome x only if x is the last ranked alternative for j. Since x is a majority winner and if j voted against x with respect to all other alternatives he cannot change the outcome. In conclusion, there is no other strategy which leads to an outcome which stochastically dominates x for j.

Consider the case where there is no majority winner, i.e., $F(\underline{P})$ is the uniform probability over outcomes. Note that no element

in the range of F stochastically dominates the uniform distribution over A except j's first ranked alternative. But if j voted for his first ranked alternative against all other alternatives, we see that no change in his preferences can lead to this first ranked alternative being a majority winner. In conclusion j cannot beneficially manipulate F at $\underline{P}$.

Q.E.D.

In the theorem of the previous section we exhibited an RCF which was Pareto undominated. In this section we have exhibited an RCF which is straightforward. We note that if there are three people and three alternatives these two RCFs coincide. Thus for three persons and three alternatives the two desiderata are compatible. However, if there are at least four people and at least four alternatives the two objectives are no longer compatible.

Theorem: If $\#N > 3$ and $\#A > 3$ then there does not exist an RCF which is simultaneously Pareto undominated and straightforward.

Proof: Consider Γ, the symmetric span of the following five person four alternative profile:

	x	x	y	y	w
(**)	y	y	x	x	z
	w	w	z	z	y
	z	z	w	w	x

Clearly, on Γ q_y stochastically dominates q_x, q_w, and q_z. Hence q_y stochastically dominates any outcome which assigns positive probability to any alternative other than y. By the lemma of the previous section any (Pareto) undominated RCF must assign y, the degenerate random outcome, to the profiles in Γ.

Now consider the symmetric span Δ of a profile which is identical to the profile (**) except that the preference ordering for the first person is (x w z y) instead of (x y w z). The induced probability vectors for Δ are $q_x = (2/5, 2/5, 0, 1/5)$, $q_y = (2/5, 1/5, 1/5, 1/5)$, $q_w = (1/5, 1/5, 1/5, 2/5)$, $q_z = (0, 1/5, 3/5, 1/5)$. Hence by the same argument above, an undominated RCF must assign x with probability one to Δ.

We have thus shown what any undominated RCF must assign to the profiles in Γ and Δ. But we see that this restriction is not compatible with straightforwardness. If the first person in the profile (**) announces the preference ordering (x w z y) instead of his correct preferences (x y w z) he changes the outcome from y to x (which he prefers to y according to his true preferences).

We will now show that efficiency and straightforwardness are incompatible for any odd number of persons greater than 3 and any number of alternatives greater than 3. Suppose we append to the profile (**) equal numbers of persons with preference orderings (x y w z) and (y x z w). Clearly the vectors q_x and q_y for the appended profile will still stochastically dominate q_w and q_z. Because of the symmetry between x and y in the appended preferences, the relation between q_x and q_y is also unchanged. Thus for an undominated RCF, y must be chosen for the expanded profile. Analogously if the first person's preferences are changed as before the undominated RCF must assign x to the new profile hence as before person 1 can beneficially misrepresent his preferences in the appended (**) profile. To see the impossibility of obtaining an undominated and straightforward RCF for a greater number of alternatives (still with

an odd number of persons > 3), consider extending the profile (**) by listing additional alternatives below the original four alternatives in each person's preference ordering. The above arguments showing the possibility of misrepresentation by person one remain essentially unchanged.

To complete the proof we must deal with the case of an even number of persons greater than 3. We call the reader's attention to the four person, four alternative profile (*) in the previous section. It is easy to verify that if an RCF is undominated person one can beneficially misrepresent his preferences in the same manner as above. It is true not only for the profile (*) but also when the profile is extended by adding more alternatives or by adding pairs of persons.

Q.E.D.

As mentioned earlier, the symmetry assumption on RCFs and ROFs is superfluous given the symmetry of the priors. In general, given any non-symmetric ROF (or RCF), there exists a symmetric ROF which induces on the symmetric span of any profile an identical probability vector. In particular, in the proof of the theorem we see that for the four symmetric spans of each of the four profiles used (*, ** and their variants), efficiency requires that the RCF will be constant. Constancy on a symmetric span implies that the RCF is symmetric on that span.

There are however, certain non-symmetric priors for which the random dictator is not Pareto dominated. Since it is always straightforward, we see why the assumed symmetric priors are essential to our result.

must be chosen for any cardinal profile which induces an ordinal profile in this symmetric span if an RCCF is to be efficient.

Then for all the 4 x 4 matrices with alternatives x, y, z, w which induce the ordinal profile (*) of section three, we see that the unique efficient, symmetric RCCF must choose for each of these matrices the degenerate random outcome which assigns probability 1 to outcome y. Similarly for each of the ordinal profiles of section three in which there was an outcome which stochastically dominated all other outcomes. Thus once we note that the appropriate cardinal analog of the lemma in section two is true, we see that the incompatibility of efficiency and straightforwardness extends to RCCFs.

However, in the previous sentence we have glossed over precisely what is entailed in an RCCF being straightforward. Given an ROF $f: S^N \to \hat{A}$ and a cardinal profile $\big(U(x,i)\big)_{x\varepsilon A,\, i\varepsilon N}$ a strategy selection $\underline{s}^* = (s_i^*)_{i\varepsilon N}$ is said to be a <u>cardinal</u> dominant strategy equilibrium if for all j in N and for all $\underline{s} \varepsilon S^N$, $u(\cdot,j)\cdot f(\underline{s}) \leq u(\cdot,j)\cdot f\big(s_j^*,(s_i)_{i\varepsilon N\setminus\{j\}}\big)$. This is nothing more than the requirement that person j's expected utility is maximized by choosing s_j^* regardless of the strategies selected by others. An ROF is cardinally straightforward if for almost every cardinal profile there is a dominant strategy equilibrium. Analogous to the ordinal case, if an RCCF can be implemented by a straightforward ROF, it can be implemented by itself with preferences as strategies. In this way our impossibility result holds for the cardinal case regardless of whether the strategy spaces of the ROFs are finite or infinite.

In the case that we restrict ourselves to finite strategy spaces for ROFs Gibbard [1978] showed that if an ROF was cardinally

IV. Some Comments on Extension to a Cardinal Framework

There are several ways to incorporate cardinality into social choice models. In all of these approaches the basic notion is that when a society is faced with a choice problem, each person has cardinal preferences over the outcomes. We mean by a person's having cardinal preferences his ability to choose between random outcomes via expected utility. A cardinal profile is thus an $\#N \times \#A$ matrix each column of which is the utility vector for an individual. A random cardinal choice function (RCCF) maps the set of such matrices into $\hat{A}$. In order to discuss the ex ante efficiency of RCCFs we assume as before that each individual has a prior probability distribution over profiles. Here, of course, the profiles are the matrices introduced above.

Since a person makes his evaluations via expected utility there is now a single number which is the expected utility of an RCCF with respect to his prior. Different persons may have different priors, thus they may attach different values to an RCCF. As before, we will assume symmetry of priors and support consisting of all matrices. Symmetry here can be defined by restricting our attention to those measures which have appropriate density functions or by introducing transformations of the space of matrices into itself induced by permutations of columns. If we are considering symmetric RCCFs (i.e., symmetry across permutations of columns) we see that for each person there is a unique (up to indifference) expected utility maximizing random outcome independent of his prior. Hence there is a unique (again up to indifference) symmetric RCCF. Suppose for the symmetric span of an ordinal profile there is an outcome which stochastically dominates all other random outcomes. Then that outcome

straightforward, it was a convex combination of dictatorial ROFs and ROFs with two element ranges. This result can be used to show the incompatibility of straightforwardness and efficiency in this special case. However, Zeckhauser [1973] noticed that you cannot expect efficiency with outcome functions which have essentially fewer strategies than the number of different environments (profiles) to which they are to be applied. We point out that our impossibility theorem does not hinge on this difficulty as we made no assumptions on the strategy spaces.

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